Homework 1

Tools of the Trade + Gravity & Orbits

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Learning Objectives

- Distinguish what astronomers **directly measure** from what we **infer** using models.
- Use **dimensional analysis** as a correctness filter (and as a tool for deriving scalings).
- Use the **ratio method** to scale solutions without re-deriving everything from scratch.
- Convert cleanly between **SI and CGS** and carry units through every step.
- Apply **Kepler's Third Law** as a scaling law.
- Compute and interpret **orbital velocity** from Newtonian gravity, including unit checks and physical trends.
- Use **energy arguments** to determine whether an object is gravitationally bound or unbound.
- Use pressure balance + dimensional reasoning to derive the qualitative **white dwarf mass–radius scaling**.

Concept Throughline

Keep these in mind on every problem:

- **Measurement → model → inference** (and where assumptions sneak in).
- **Units are physics.** If the units don't work, the equation doesn't work.
- **Scaling laws beat big numbers.** Ratios are your friend.
- **Sanity checks are required.** Physics answers should make physical sense.

Prerequisites

- Scientific notation; exponent rules
- Basic algebra (solve for a variable; proportional reasoning)
- Comfort with units and unit conversion (especially cm, m, km; g, kg; s)

Relevant Readings

- Lecture 1: Spoiler Alerts
- Lecture 2: Tools of the Trade
- Lecture 3: Gravity and Orbits

Note

Before you start: Review the Homework Guidelines for required format and tools.

Tip

HW1 only: Each problem below includes a "Tool(s)" hint to help you practice identifying which method to use. Future assignments will not include these hints—you'll need to recognize which tools apply on your own. Use your lecture notes; that's the point.

Tip

HW1 Note: Solutions will be posted Wednesday morning. We'll go over how to write the Grade Memo in class on Thursday.

Problems (10 total)

Problem 1 — Observable vs. Model-Dependent

Tool(s) you should use: Conceptual/model reasoning (Measurement → model → inference)

A news article claims "Astronomers measured the temperature of TRAPPIST-1e to be 255 K." Critique this statement. What was actually measured, and what model was used to arrive at the temperature?

Problem 2 — Empirical vs. Physical

Tool(s) you should use: Conceptual/model reasoning

Lecture 1 introduced the course thesis: *pretty pictures* → *measurements* → *models* → *inferences*.

- (a) What is the difference between an empirical law and a physical law?
- (b) Give one example from the lectures where an empirical pattern is explained by a physical mechanism.
- (c) Why does this distinction matter for how we do science?

Problem 3 — Dimensional Analysis Check

Tool(s) you should use: Dimensional analysis

For each equation below, determine whether it is dimensionally valid. If not, identify what's wrong.

- (a) Orbital velocity: $v = \sqrt{GM/r}$
- (b) Kinetic energy: $E = mv$
- (c) Gravitational potential energy: $U = GMm/r^2$
- (d) Pressure: $P = \rho v^2$ (where ρ is density)

Problem 4 — The Dimensions of σ

Tool(s) you should use: Dimensional analysis + unit conversion

The Stefan-Boltzmann law relates a star's luminosity to its radius and surface temperature:

$$L = 4\pi R^2 \sigma T^4$$

where L is luminosity (energy per time), R is radius, T is temperature, and σ is the Stefan-Boltzmann constant.

- (a) Derive the dimensions of σ in terms of $[M]$, $[L]$, $[T]$, and $[\Theta]$ (temperature).

  Hint: Luminosity is the radiative energy emitted per unit time.

- (b) Verify that the CGS unit of σ ($\mathrm{erg\,s^{-1}\,cm^{-2}\,K^{-4}}$) matches these dimensions.

- (c) The SI value of σ is $5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$. Convert this to CGS units.

  Hint: $1\,\mathrm{W} = 10^7\,\mathrm{erg/s}$ and $1\,\mathrm{m} = 100\,\mathrm{cm}$.

Problem 5 — Kepler Scaling

Tool(s) you should use: Ratio method + sanity check

Using Kepler's third law (with P in years and a in AU for objects orbiting the Sun):

- (a) Neptune orbits at 30 AU. Estimate its orbital period.
- (b) An asteroid has orbital period 8 years. Estimate its semi-major axis.
- (c) A Kuiper Belt object at 40 AU has what orbital period?
- (d) **Sanity check:** Pluto orbits at ~40 AU with period ~248 years. Does your answer to (c) agree?

Problem 6 — Fermi Estimation: Light-Minutes

Tool(s) you should use: Order-of-magnitude (Fermi) + unit conversion

Estimate without a calculator:

- (a) Light travels at 3×10^{10} cm/s = 3×10^{5} km/s. How far does light travel in 1 second? In 1 minute?
- (b) The Earth-Sun distance is ~1.5×10^{8} km. How many light-minutes is this?
- (c) Jupiter is ~5 AU from the Sun. How many light-minutes is this?

Problem 7 — The Centripetal Force Misconception

Tool(s) you should use: Conceptual/model reasoning

A student says: "A planet in orbit experiences two forces: gravity pulling it toward the Sun, and centripetal force keeping it in a circle."

- (a) What is wrong with this statement?
- (b) Rewrite the statement correctly.
- (c) If someone mentions "centrifugal force," under what circumstances (if any) would that be valid?

Problem 8 — Exoplanet Orbital Velocities

Tool(s) you should use: Ratio method + sanity check

Use the ratio method with Earth as your reference ($M = M_\odot$, $r = 1$ AU). Express answers in terms of $v_\oplus$.

- (a) A "hot Jupiter" orbits a Sun-like star at $r = 0.05$ AU. Find $v/v_\oplus$.
- (b) A planet orbits a red dwarf ($M = 0.25\,M_\odot$) at $r = 0.25$ AU. Find $v/v_\oplus$.
- (c) **Sanity check:** Do your answers to (a) and (b) make physical sense? Explain.

Problem 9 — Bound or Unbound?

Tool(s) you should use: Energy reasoning + sanity check

A comet passes through Earth's orbit (1 AU from the Sun) traveling at 50 km/s.

- (a) Will the comet escape the Solar System, or is it gravitationally bound?
- (b) **Sanity check:** How does your answer relate to Earth's orbital velocity?

Problem 10 — Why White Dwarfs Get Smaller When They Gain Mass

Tool(s) you should use: Dimensional analysis + model reasoning + sanity checks

> ℹ Note
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> **Heads up:** This is the capstone problem for HW1—it integrates dimensional analysis, scaling, and model reasoning across multiple parts. Budget extra time compared to the other problems.

This multi-part problem guides you through one of the most counter-intuitive results in astrophysics: **more massive white dwarfs are smaller**. You'll derive this using only dimensional analysis and pressure balance—the same toolkit from lecture.

Background: A white dwarf is a stellar remnant supported not by thermal pressure (like the Sun), but by *electron degeneracy pressure*—a quantum mechanical effect. When matter is compressed to extreme densities, electrons resist being squeezed into the same quantum state (the Pauli exclusion principle). This creates pressure even at zero temperature.

The new ingredient: Quantum mechanics introduces Planck's constant $\hbar$ (pronounced "h-bar"):

$$[\hbar] = [ML^2T^{-1}]$$

This has dimensions of angular momentum. When $\hbar$ appears in an equation, you know quantum mechanics is at play.

Part A: Degeneracy Pressure from Dimensional Analysis

The degeneracy pressure P_{deg} in a white dwarf depends on:

- Planck's constant $\hbar$ with $[\hbar] = [ML^2T^{-1}]$
- Electron mass m_e with $[m_e] = [M]$
- Number density of electrons n with $[n] = [L^{-3}]$ (electrons per volume)

Note: Temperature does NOT appear—degeneracy pressure is independent of temperature!

1. What are the dimensions of pressure? Express $[P]$ in terms of $[M]$, $[L]$, $[T]$.

   Hint: Pressure is force per unit area: $[P] = [F]/[A]$. You know the dimensions of force from $F = ma$.

2. Assume $P_{deg} \propto \hbar^a m_e^b n^c$. Set up the dimensional equations by matching exponents.

3. Solve for a, b, and c.

4. You should find:

$$P_{deg} \propto \frac{\hbar^2}{m_e} n^{5/3}$$

   Verify that your exponents from step 3 give this result.

5. **Interpretation:** Why does it make physical sense that pressure increases with electron number density? What does the exponent 5/3 suggest (qualitatively) about how quickly the pressure rises?

Part B: Linking Density to Mass and Radius

Assume a white dwarf is approximately a sphere of radius R and total mass M.

(i) The average mass density is $\rho \sim M/R^3$. Write this relation and check the dimensions.

(ii) The electron number density n is proportional to mass density: $n \propto \rho/m_p$ where m_p is the proton mass (a constant). For scaling purposes, treat m_p as a constant and write:

$$n \propto \frac{M}{R^3}$$

Part C: Gravitational Pressure

Gravity tries to compress the star. The inward "gravitational pressure" scale is:

$$P_{grav} \sim \frac{GM^2}{R^4}$$

(i) Use dimensional analysis to verify that GM^2/R^4 has units of pressure.

(ii) **Physical interpretation:** Why does gravitational pressure scale like M^2? Why does it depend so strongly on R (as R^{-4})?

Part D: Pressure Balance and the Mass–Radius Relation

A stable white dwarf exists when degeneracy pressure balances gravity:

$$P_{deg} \sim P_{grav}$$

(i) Substitute your scaling for P_{deg} from Part A and your scaling for n from Part B into the pressure balance condition.

(ii) Solve for how radius R scales with mass M. You should find:

$$R \propto M^{-1/3}$$

(iii) **Interpretation:** Explain in words why adding mass makes the radius smaller in this regime. (This is the main conceptual takeaway.)

Part E: Numerical Scaling

We can turn the scaling into a rough numerical estimate.

Suppose a typical white dwarf has:

- Mass $M_0 = 0.6\,M_\odot$
- Radius $R_0 = 1\,R_\oplus$ (about Earth-sized)

(i) Use $R \propto M^{-1/3}$ to estimate the radius of a white dwarf with $M = 1.2\,M_\odot$ (twice as massive). Express your answer in Earth radii.

(ii) The Chandrasekhar limit is about $1.4\,M_\odot$—the maximum mass a white dwarf can have before it collapses. As $M \to 1.4\,M_\odot$, what is happening to R according to your scaling law?

(iii) **Limitation:** Your derivation assumed *non-relativistic* electrons. Near the Chandrasekhar limit, electrons move at speeds approaching c, and relativistic effects change the pressure-density relation. What do you think happens to the $R \propto M^{-1/3}$ scaling when electrons become relativistic? (Qualitative answer only.)

Part F: Synthesis and Reflection

(i) List all the physical ingredients that went into this derivation: which constants, which scaling laws, which physical principles?

(ii) At no point did we solve a differential equation or use calculus. Yet we derived one of the most important results in stellar astrophysics. What does this tell you about the power of dimensional analysis?

(iii) **Connection to the course thesis:** In this problem, what did we *measure* (or take as given), what *model* did we use, and what did we *infer*?

Note: Solutions will be provided separately.

Grading

Homework is graded on the **0–5 scale** described in the syllabus. The score reflects **visible reasoning and honest effort**, not just final answers.

What earns credit:

- Showing your steps (not just final numbers)
- Carrying units through calculations
- Using the indicated tool(s) explicitly
- Including sanity checks
- Clear, readable work

What loses credit:

- Missing reasoning ("I got 42" with no work shown)
- Missing or inconsistent units
- No sanity check on quantitative results
- Submitting work that *looks* like effort but doesn't make physical sense

Bottom line: A wrong answer with clear reasoning beats a right answer with no work. Nonsense dressed up as effort earns a low score.

See the syllabus for the full rubric and grade memo requirements.