Homework 8

Stellar Structure II — Energy Transport, Mass Limits, and White Dwarfs (Exam 2 Prep)

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Learning Objectives

- Use opacity, optical depth, mean free path, and random-walk reasoning to explain why energy escapes stars slowly.
- Apply toy stellar-structure scalings to connect mass to luminosity, radius, density, and lifetime.
- Explain why quantum mechanics sets the minimum stellar mass while radiation pressure helps set the upper end.
- Trace the low-mass evolutionary path from core hydrogen exhaustion to red giant, helium burning, planetary nebula, and white dwarf.
- Explain how electron degeneracy pressure supports white dwarfs and why the Chandrasekhar limit imposes a maximum white-dwarf mass.

Concept Throughline

- **Tiny photon steps make energy transport slow even when photons move locally at c.**
- **Stellar structure is a coupled balance problem, so scaling relations reveal real physics instead of just producing numbers.**
- **The stellar mass range is bounded from below by degeneracy and from above by radiation pressure and winds.**
- **Low-mass stellar death exposes a degenerate core rather than triggering core collapse.**
- **White dwarfs are supported by quantum mechanics, but only up to the Chandrasekhar limit.**

Prerequisites

- Homework 7 tools: scaling, dimensional analysis, hydrostatic reasoning, and sanity checks
- Module 3 Readings 4–8
- Scientific notation and algebra with power laws
- Clear observable → model → inference reasoning in complete sentences

Relevant Sources (Module-Based)

- Module 3 (reading): Lecture 4 — The Long Way Out: Radiation Transport
- Module 3 (reading): Lecture 5 — The Stellar Blueprint: Structure Equations and Main-Sequence Scalings
- Module 3 (reading): Lecture 6 — Nature's Narrow Window: Minimum and Maximum Stellar Masses
- Module 3 (reading): Lecture 7 — The Gentle Death of a Sun-Like Star: Low-Mass Stellar Evolution
- Module 3 (reading): Lecture 8 — The Quantum Limit: Degeneracy Pressure and the Chandrasekhar Mass

> **Note**
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> **Before you start:** Review the Homework Guidelines for required format and tools.

> **Tip**
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> **HW8 note:** Treat this like a guided study set: show the units, say what each scaling assumes, and include a one-line physical interpretation after every major result.

> **Note**
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> Use these values and relations unless a problem states otherwise:
> These toy stellar-structure scalings are for **roughly solar-like, radiative pp-chain main-sequence stars**. Use them only when a problem explicitly tells you to do so, and say that assumption out loud in your solution.
>
> - Mean free path:
>
> $$\ell = \frac{1}{\kappa\rho}$$
>
> - Optical depth:
>
> $$\tau \sim \kappa\rho R \sim \frac{R}{\ell}$$
>
> - Diffusion-time estimate:
>
> $$t_{\rm diff} \sim \frac{R^2}{\ell c}$$
>
> - Toy main-sequence scalings for radiative pp-chain stars:
>
> $$\frac{L}{L_\odot} \approx \left(\frac{M}{M_\odot}\right)^3, \qquad \frac{R}{R_\odot} \approx \left(\frac{M}{M_\odot}\right)^{3/7}$$
>
> $$\frac{t_{\rm MS}}{t_\odot} \approx \left(\frac{M}{M_\odot}\right)\left(\frac{L}{L_\odot}\right)^{-1}, \qquad t_\odot \approx 10 \text{ Gyr}$$
>
> - Eddington luminosity for electron-scattering opacity:
>
> $$L_{\rm Edd} = \frac{4\pi G M c}{\kappa} \approx 3.8\times 10^4 \left(\frac{M}{M_\odot}\right) L_\odot$$

- White-dwarf mass-radius trend (non-relativistic regime):

$$R \propto M^{-1/3}$$

- White-dwarf average density:

$$\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3}$$

- Useful constants:

$$c = 3.0 \times 10^{10}\ \text{cm/s}, \quad M_\odot = 2.0 \times 10^{33}\ \text{g}, \quad R_\odot = 7.0 \times 10^{10}\ \text{cm}, \quad R_\oplus = 6.4 \times 10^{8}\ \text{cm}$$

- Reference scales:

$$\kappa_{\text{es}} \approx 0.34\ \text{cm}^2\ \text{g}^{-1}, \quad \rho_{\text{core},\odot} \approx 150\ \text{g/cm}^3, \quad M_{\text{Ch}} \approx 1.4\, M_\odot$$

Note

Required reporting format:

- Every numeric answer must include units.
- Show the identity-trick logic for unit conversions instead of skipping straight to a converted number.
- Include a one-line sanity check for each calculation problem.
- For conceptual and synthesis problems, write in complete sentences and name the key physical idea explicitly.
- When you use a scaling relation, say what assumption makes that scaling reasonable.
- When a problem asks for a causal explanation, write it as a short because → therefore chain rather than a list of disconnected facts.

This homework is one connected scientific story. Energy released in the core does not escape freely, so transport physics helps determine stellar structure. Structure and radiation then help explain why stars occupy only a narrow mass range. Once a low-mass star exhausts core hydrogen, that same structure-and-transport logic drives it toward a white dwarf, and the observed masses and radii of white dwarfs let us infer where quantum support finally fails.

Problems (9 total)

Part A — Why Energy Escapes So Slowly

Problem 1 — Why the Sun Is Not a Transparent Bulb

A student says, “Photons move at the speed of light, so energy made in the solar core should reach the surface in a few seconds.”

- (a) Explain the missing physics in this statement.
- (b) In words, distinguish **opacity** κ from **optical depth** τ.
- (c) Explain why, in stellar interiors, what diffuses outward is the **energy of the radiation field** rather than the identity of one specific photon.

Problem 2 — Mean Free Path, Optical Depth, and Diffusion Time

Use a one-zone solar estimate with

- $\kappa = 0.34\ \mathrm{cm^2\,g^{-1}}$
- $\rho = 150\ \mathrm{g/cm^3}$
- $R = R_\odot = 7.0 \times 10^{10}\ \mathrm{cm}$
- (a) Compute the photon mean free path ℓ in cm.
- (b) Compute the optical depth τ across one solar radius.
- (c) Estimate the photon diffusion time t_{diff} in seconds and in years.
- (d) Compare your result to the straight-line light-crossing time $t_{\mathrm{stream}} = R_\odot/c$, and explain in one sentence why the two answers differ so dramatically.

Problem 3 — Why Different Stars Transport Energy Differently

Compare a **solar-like** main-sequence star with a **high-mass** main-sequence star. (Do not use the very-low-mass, fully convective regime for this problem.)

- (a) Explain why a **high-mass** star tends to develop a **convective core**.
- (b) Explain why a **solar-like** star can instead have a **radiative core** and a **convective envelope**.
- (c) In 3–5 sentences, connect these transport regimes to the deeper idea that energy transport depends on whether radiation can carry the required luminosity efficiently enough.

Part B — Main-Sequence Scalings and Mass Limits

Problem 4 — Toy Main-Sequence Scalings in Their Proper Regime

Use the toy scalings

$$\frac{L}{L_\odot} \approx \left(\frac{M}{M_\odot}\right)^3, \qquad \frac{R}{R_\odot} \approx \left(\frac{M}{M_\odot}\right)^{3/7}, \qquad \frac{t_{\rm MS}}{t_\odot} \approx \left(\frac{M}{M_\odot}\right)\left(\frac{L}{L_\odot}\right)^{-1}.$$

For a **toy radiative, pp-chain main-sequence star** with $M = 1.1\, M_\odot$:

- (a) Estimate its luminosity in solar units.
- (b) Estimate its radius in solar units.
- (c) Use $\bar{\rho} \propto M/R^3$ to compute its **average density relative to the Sun**.
- (d) Estimate its main-sequence lifetime in Gyr.
- (e) State clearly why this problem is a **toy-model exercise** rather than a full description of every main-sequence star.

Problem 5 — Why Nature Allows Only a Narrow Stellar Mass Range

Hydrogen-burning stars occupy only a limited mass range, and astronomers really do observe both substellar brown dwarfs and a very sparse extreme upper-mass tail.

- (a) Explain why objects below about $0.08\, M_\odot$ become **brown dwarfs** instead of sustained hydrogen-burning stars.
- (b) Explain why stars near the upper end of the mass range run into the **Eddington regime** and strong mass loss.
- (c) In 2–4 sentences, explain why the lower and upper limits come from **different** physical mechanisms and how that physics helps explain the observed stellar population.

Problem 6 — Estimating the Maximum Stellar Mass

Set the ordinary main-sequence luminosity scaling

$$\frac{L}{L_\odot} \approx \left(\frac{M}{M_\odot}\right)^{3.5}$$

equal to the Eddington scaling

$$\frac{L_{\rm Edd}}{L_\odot} \approx 3.8 \times 10^4 \left(\frac{M}{M_\odot}\right).$$

- (a) Solve for the characteristic mass scale where the two become comparable.

- (b) Report your answer as an **order-of-magnitude upper-mass scale** in solar masses, not as a fake-precision exact cutoff.
- (c) Explain why this is an **order-of-magnitude ceiling** rather than a hard wall at exactly one mass.
- (d) Observations show stars up to roughly 150–200 $M_\odot$. Explain why that does **not** contradict your estimate.

Part C — Low-Mass Evolution and White Dwarfs

Problem 7 — From Main Sequence to Red Giant

Build the post-main-sequence evolution of a low-mass star as a **causal chain** after core hydrogen is exhausted.

- (a) What changes first in the **core**, and what becomes the star's dominant energy source once core hydrogen burning stops?
- (b) How does the **core** respond to that change, and how does that help strengthen hydrogen-shell burning?
- (c) How does the **envelope** respond, and why does this produce a **red giant** rather than simply a smaller, hotter star?
- (d) Why does **degeneracy** matter for helium ignition in low-mass stars, and how do the later AGB + mass-loss stages expose a white dwarf?

Problem 8 — White Dwarf Density and the Reversed Mass-Radius Relation

A typical white dwarf has mass $M = 0.6\,M_\odot$ and radius $R = R_\oplus$.

- (a) Compute its average density in g/cm^3.
- (b) Compare your result to Earth's average density, $\bar{\rho}_\oplus \approx 5.5$ g/cm^3.
- (c) A second white dwarf has mass $1.2\,M_\odot$. Using $R \propto M^{-1/3}$, compute the radius ratio $R_{1.2}/R_{0.6}$.
- (d) In 2–4 sentences, explain physically why the more massive white dwarf is **smaller**, not larger, and state what mass–radius trend astronomers should therefore expect to infer from white dwarf observations.

Problem 9 — The Chandrasekhar Limit as an Inference Chain

Astronomers observe white dwarfs in binaries and infer their masses and radii. Stable white dwarfs are compact, Earth-sized, degenerate objects, and the stable population cuts off near $1.4\,M_\odot$.

- (a) State the **observable**, the **model**, and the **inference** in this story.
- (b) Explain why **electron degeneracy pressure**, rooted in the Pauli exclusion principle, can support a white dwarf even when there is no fusion.
- (c) Explain why relativity changes the game as the white dwarf mass rises toward the Chandrasekhar limit.
- (d) In 3–5 sentences, explain what the Chandrasekhar limit tells us about the fate of stellar cores above $1.4\,M_\odot$.